

Global Duality, Signature Calculus and the Discrete Logarithm Problem

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ABSTRACT. We study the discrete logarithm problem for the multiplicative group and for elliptic curves over a finite field by using a lifting of the corresponding object to an algebraic number field and global duality. We introduce the *signature* of a Dirichlet character (in the multiplicative group case) or principal homogeneous space (in the elliptic curve case), which is a measure of the ramification at certain places. We then develop *signature calculus*, which generalizes and refines the index calculus method. Finally, we show the random polynomial time equivalence for these two cases between the problem of computing signatures and the discrete logarithm problem.

AMS Subject Classification: 11G05, 11R37, 11Y40 (primary), 14G50, 68W20 (secondary)

1. Introduction

Let A be a finite abelian group, which we write additively, and let x be an element of A . Let y be in the subgroup generated by x , so that $y = nx$ for some positive integer n . Recall that the *discrete logarithm problem* (DLP) is to determine n in a computationally efficient way. The computational complexity of this problem when the bit size of the inputs is large is the basis of many public-key encryption schemes used today. Two of the most important examples of finite abelian groups that are used in public-key cryptography are the multiplicative group of a finite field and the group of points on an elliptic curve over a finite field (see [Ko] and [Mill] for the original papers and [KMV] for a survey of work as of 2000).

In what follows below, we will assume that ℓ is a large prime number dividing the order of A and that x is an element of order ℓ . For p a prime number and q a power of p , we denote by \mathbb{F}_q the finite field with q elements and by \mathbb{F}_q^* its multiplicative group of nonzero elements.

One of the best-known techniques to address the DLP is *index calculus*, which uses relations between elements of an abelian algebraic group to derive linear relations between their discrete logarithms. In the case of the multiplicative group of a finite prime field, \mathbb{F}_p , taking sufficiently many random liftings of elements of \mathbb{F}_p^* to integers will ensure that some will only be divisible by small (compared to p) prime numbers. Then such relations can be derived because we know how to efficiently factor integers that are products of powers of small prime numbers. See e.g. [Mc], §5.1 or [SWD] for details. Trying to imitate this method for an elliptic curve by lifting the curve to an algebraic number field has turned out to be less effective, because the behavior of the height function on the Mordell-Weil group of the lifted curve makes it far more difficult to derive relations like those just mentioned in the multiplicative group case (see [HKT] or [JKSST] for more details). However an important aspect of index calculus has not been addressed in these studies, namely, the idea of leveraging small primes to tackle a computational problem that involves large primes, and it is not clear how this idea can be put to work in a setting that involves the Mordell-Weil groups of elliptic curves. In this paper we address this issue in both cases from the perspective of arithmetic duality and propose a unified method which we call *signature calculus*.

Our general strategy to address the DLP in an abelian algebraic group is to take a lifting of the group to an algebraic number field and use the reciprocity law of global class field theory. Others have taken this approach (see e.g. [F] [FR], [N]), and we refine their methods and give a general exposition of the theory. We explain below in detail how this works for the multiplicative group of a finite field and for the group of points of an elliptic curve over a finite field. The idea is to construct a suitable “test” element, which is a Dirichlet character in the multiplicative group case and a principal homogeneous space in the elliptic curve case. This element pairs with the lifting of a point of the group to give an equation between the local terms of this pairing. The lifting from a finite field \mathbb{F}_p to a global field preserves discrete logarithms at a place over p . The reciprocity law then allows us to distribute information on the discrete logarithms among a set of places which depends on the choice of test element and the manner of lifting. We define the *signature* of these test elements and prove the equivalence of computing the signature with the respective DLP. These signatures measure the ramification at primes above p and ℓ . Though the signatures are small, they uniquely identify the objects they represent (Dirichlet characters and principal homogeneous spaces). They are, in fact, succinct representations of those objects, and the equivalence results show that computing these signatures (without constructing the objects they succinctly represent) amounts to solving discrete-log problems.

The unifying approach based on global duality provides an ideal setting to compare and contrast index calculus methods in the multiplicative group and elliptic curve cases. The signature computation problem involves large primes, and the question naturally arises as to whether small primes can be utilized to tackle the problem with greater computational efficiency, in a similar way as we mentioned for the multiplicative group. Following the equivalence results we show that in this setting, the index calculus method arises quite naturally for the discrete-log problem in the multiplicative case and the corresponding signature computation

problem. In contrast, a similar method cannot be fashioned for the elliptic curve case. The success in one case and the lack thereof in the other is due to the difference in the nature of the pairings involved. In the multiplicative case, a Dirichlet character which is unramified at a finite place v can nevertheless pair nontrivially with local non-units at v . This makes it possible for small primes to play a role in forming relations among values of local pairings. In the elliptic curve case, an unramified principal homogeneous space at a good reduction place v is one that extends to a principal homogeneous space under a smooth proper model \mathcal{E}_v of E over the ring of local integers R_v (please see §1 below for more details and explanation). There is a bijection between such principal homogeneous spaces and the corresponding objects under the reduction of $\mathcal{E}_v \bmod v$ (see e.g. [MET], Chapter III, Remark 3.11(a)). By a theorem of Lang ([L], Theorem 2), the latter objects are trivial. Thus, in the elliptic curve case, an unramified principal homogeneous space at a good reduction place is trivial. For small primes of bad reduction not dividing ℓ , only the group of components of the special fibre of the Néron model of the elliptic curve over the ring of integers plays a role, and the order of this group is unlikely to be divisible by ℓ (see §5.1.2 below for more details). As a result, only primes of large norm can play a role in forming relations among values of local pairings in the elliptic curve case.

The computational complexity of signature calculus is an intriguing question, since the objects involved (Dirichlet characters and principal homogeneous spaces) and their associated field extensions are huge, but the signatures sought are small. Although we show that the testing Dirichlet characters and principal homogeneous spaces exist, it remains an interesting question as to how they can be explicitly constructed. This is easier to handle in the multiplicative case, where we also derive a concrete number theoretic characterization of the character signature by working out the local pairings using norm residue symbols. For the elliptic curve case, we have a partial solution.

This paper is a more formal and detailed exposition than the survey of this material that appeared in [HRANTS], and it contains very significant material that is not in that paper. We have tried to be completely mathematically precise while retaining the cryptographic motivation and applications.

The idea of using global methods in this way was originally proposed by Frey [F], whom we thank for inspiration, helpful discussions, and for inviting us to present our work at the Elliptic Curve Cryptography (ECC) conference in Bochum in September 2004. Methods of this type have also been used by Frey and Rück [FR], and by Nguyen [N].

2. Global Framework

2.1. Notation and Preliminaries. If A is a locally compact abelian group that is either profinite or torsion, we denote by A^* the group $\text{Hom}_{\text{cont}}(A, \mathbb{Q}/\mathbb{Z})$ of continuous homomorphisms and refer to it as the *Pontryagin dual* of A . Note that $*$ is an exact functor since \mathbb{Q}/\mathbb{Z} is a divisible abelian group.

Let K be a field, fix a separable closure \overline{K} of K , and let $G = \text{Gal}(\overline{K}/K)$. Let M be a discrete G -module upon which G acts continuously, where G has the Krull topology. We will be using Galois cohomology extensively, which we will denote by $H^i(G, M)$ or sometimes $H^i(K, M)$. A basic reference for this theory is [S1].

We shall mainly be using three types of fields: finite fields, denoted by \mathbb{F} , algebraic number fields, denoted by K , and the completion of an algebraic number field at a finite place v , denoted by K_v .

An *algebraic number field* will be a finite extension of the field of rational numbers \mathbb{Q} . We consider equivalence classes of absolute values v on K , which we call *places*. As most of our discussion will pertain to abelian groups that are ℓ -torsion, where ℓ is an odd prime number, we shall ignore the real and complex places for the most part.

Let R be a discrete valuation ring with fraction field K and residue field F . For example, R could be the ring of integers in a K_v . Let X be a smooth proper scheme over $Y = \text{Spec}(R)$. Recall that this means that the structure morphism:

$$f : X \rightarrow Y$$

is smooth and proper. The former condition means that the fibres over K (the generic fibre) and F (the special fibre) are smooth, and the latter means that f is separated and universally closed (i.e. that if we change base by a morphism $Z \rightarrow Y$, then the morphism:

$$X \times_Y Z \rightarrow Z$$

is closed). If $X \rightarrow Y$ is a proper morphism, then a point $P \in X(K)$ may be lifted to a point in $X(R)$. If E is an elliptic curve over K , we may clear the denominators in a defining equation and view it as a curve over R (not necessarily smooth over R). Then E is proper over R , whereas the multiplicative group is affine and decidedly not proper.

Recall that an *elliptic curve* over a field K is a smooth, projective algebraic curve E of genus 1 together with a distinguished rational point O , which serves as the identity element in an abelian group structure on E that can be defined geometrically by a chord and tangent method. We denote by $E(K)$ the set of points of E over K . Recall that a *principal homogeneous space* under E over K is a curve F of genus 1 over K together with a simply transitive group action of E on F . The isomorphism classes of such principal homogeneous spaces are classified by the group $H^1(G, E(\overline{K}))$, where $G = \text{Gal}(\overline{K}/K)$. A principal homogeneous space is trivial if and only if it has a rational point over K , in which case it is isomorphic to E over K . Thus any principal homogeneous space becomes isomorphic to E over a finite extension of K .

Let \mathcal{M} be an algebraic group over a discrete valuation ring R and denote by M its fibre over the quotient field K . We will be most interested in the cases where \mathcal{M} is either the constant algebraic group $\mathbb{Z}/\ell\mathbb{Z}$ or a smooth proper model of an elliptic curve E with good reduction over a completion of an algebraic number field at a

finite place, v . Recall that an element of $H^1(K, M)$ is said to be *unramified* if it is in the image of the natural map:

$$H^1(R, M) \rightarrow H^1(G, M).$$

This is a more general notion of non-ramification, which is the same as the usual definition when M is finite.

Let \tilde{E} be an elliptic curve over \mathbb{F} and let R be a discrete valuation ring R with quotient field K and residue field \mathbb{F} . Then a *lifting* E of \tilde{E} to K is a smooth proper scheme \mathcal{E} over R whose special fibre is \tilde{E} and whose generic fibre is E . We shall use rather simple liftings below, but let us point out that it is a theorem of Deuring [D] that if \tilde{E} is an elliptic curve over a finite field with an endomorphism φ , then the pair (\tilde{E}, φ) can be lifted to a discrete valuation ring R whose quotient field is an algebraic number field. If the curve is *ordinary*, as are the curves we consider in this paper, then one can lift the curve together with the whole endomorphism ring. A more systematic approach to liftings of ordinary elliptic curves is given by Serre-Tate theory (see e.g. [S2], §5).

Recall the Brauer group $Br(K)$ of similarity classes of finite dimensional central simple algebras over K , which can be described in terms of Galois cohomology by

$$Br(K) \cong H^2(G, \overline{K}^*).$$

When K is an algebraic number field, we have the Brauer-Hasse-Noether exact sequence:

$$(\dagger) \quad 0 \rightarrow Br(K) \rightarrow \sum_v Br(K_v) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

This is the beginning of the theory of *global duality*, which shows how to relate the arithmetic of K with that of all of the K_v . The following subsections review this theory briefly in the context in which we shall use it.

2.2. Reciprocity Law for the Multiplicative Group. We review the reciprocity law in this context, mostly following the exposition of ([S1], Chapter XIV). Let K^* denote the set of nonzero elements of K , which is an abelian group under multiplication. We consider a Dirichlet character χ of K , which we view as an element of the Galois cohomology group $H^1(G, \mathbb{Q}/\mathbb{Z})$. Thus χ represents a finite cyclic extension L/K together with a homomorphism:

$$Gal(L/K) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

Let $\partial(\chi)$ denote the image of χ under the boundary map

$$H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} H^2(G, \mathbb{Z})$$

in the long exact cohomology sequence associated to the short exact sequence of G -modules with trivial action:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

Then for $a \in K^*$ we consider

$$\langle \chi, a \rangle := a \cup \partial(\chi) \in H^2(G, \overline{K}^*)$$

under the pairing:

$$K^* = H^0(G, \overline{K}^*) \times H^2(G, \mathbb{Z}) \rightarrow H^2(G, \overline{K}^*) \cong Br(K).$$

If L is the extension corresponding to χ , then we have that $\langle \chi, a \rangle = 0$ if and only if a is a norm from L^* .

If K is an algebraic number field, $\chi \in H^1(G, \mathbb{Q}/\mathbb{Z})$, $a \in K^*$ and v is a place of K , then we can restrict χ to each K_v and regard a as an element of K_v^* . Note that we may have $\chi_v = 0$. We then denote the local pairing by $\langle \chi_v, a_v \rangle$. If v is a nonarchimedean place then $Br(K_v) \cong \mathbb{Q}/\mathbb{Z}$ and we view $\langle \chi_v, a_v \rangle$ as an element of \mathbb{Q}/\mathbb{Z} . Note also that if v is a place where χ is unramified and a is a unit at v , then $\langle \chi_v, a_v \rangle = 0$. That is, every unit is a norm from an unramified extension of nonarchimedean local fields. Thus $\langle \chi_v, a_v \rangle = 0$ for all but finitely many v . Since the local pairings are compatible with the global pairings, the exact sequence (\dagger) above for the Brauer group of an algebraic number field shows that we have the *reciprocity law*

$$\sum_v \langle \chi_v, a_v \rangle = 0 \in \mathbb{Q}/\mathbb{Z}.$$

2.3. Reciprocity Law for Elliptic Curves. Let E be an elliptic curve over K . Let $Q \in E(K)$ and $\alpha \in H^1(K, E)$. We consider the pairings

$$\begin{aligned} \langle \alpha, Q \rangle &\in Br(K) \\ \langle \alpha_v, Q_v \rangle &\in Br(K_v) \cong \mathbb{Q}/\mathbb{Z}. \end{aligned}$$

These are not as easy to describe explicitly as in the case of the multiplicative group, but we give here a quick if somewhat terse definition. Given an abelian variety A over K , let \hat{A} denote its dual, which is $Ext_K^1(A, \mathbb{G}_m)$, where \mathbb{G}_m is the multiplicative group scheme and the Ext is taken in the category of algebraic groups over K . An elliptic curve is self-dual, so that we can identify $E(K)$ with $Ext_K^1(E, \mathbb{G}_m)$. Given $Q \in E(K)$, represent it as a 1-extension of algebraic groups using this identification

$$0 \rightarrow \mathbb{G}_m \rightarrow X \rightarrow E \rightarrow 0,$$

and let

$$(\dagger\dagger) \quad 0 \rightarrow \overline{K}^* \rightarrow X(\overline{K}) \rightarrow E(\overline{K}) \rightarrow 0$$

be the short exact sequence of \overline{K} -points of these groups. Then given an element $\alpha \in H^1(G, E(\overline{K}))$, let $\langle \alpha, Q \rangle = \partial_Q(\alpha)$, the image of α under the boundary map:

$$H^1(G, E(\overline{K})) \xrightarrow{\partial_Q} H^2(G, \overline{K}^*)$$

in the long exact cohomology sequence obtained from the short exact sequence $(\dagger\dagger)$. For $\alpha \in H^1(G, E(\overline{K}))$ and $Q \in E(K)$ we denote by α_v the image of α in $H^1(G_v, E(\overline{K}_v))$ (which may be zero) and by Q_v the image of Q in $E(K_v)$. We can make a similar definition over the nonarchimedean fields K_v for $\alpha_v \in$

$H^1(G_v, E(\overline{K}_v))$ and $Q_v \in E(K_v)$ to get $\langle \alpha_v, Q_v \rangle \in Br(K_v) \cong \mathbb{Q}/\mathbb{Z}$.

We will be interested in the situation where $\alpha \in H^1(K, E)[\ell]$, in which case we have the following commutative diagram:

$$\begin{array}{ccccc} E(K)/\ell & \times & H^1(K, E)[\ell] & \rightarrow & Br(K)[\ell] \\ \downarrow & & \downarrow & & \downarrow \\ E(K_v)/\ell & \times & H^1(K_v, E)[\ell] & \rightarrow & Br(K_v)[\ell] \end{array}$$

In the case of the local field K_v , the pairing is perfect (local duality for abelian varieties, see e.g. [MAD], Ch. I, §3, Corollary 3.4).

We then have that $\langle \alpha_v, Q_v \rangle = 0$ for almost all v . The fundamental sequence (\dagger), the identification $Br(K_v)[\ell] \cong \mathbb{Z}/\ell\mathbb{Z}$, and the commutative diagram above imply that for $\alpha \in H^1(K, E)[\ell]$ and $Q \in E(K)$,

$$\sum_v \langle \alpha_v, Q_v \rangle = 0 \in \mathbb{Q}/\mathbb{Z}.$$

2.4. Cohomological Basis of the Unified Approach. Our approach is based on duality theorems for Galois modules and for abelian varieties over number fields. Let K be an algebraic number field and \mathcal{O}_K the ring of integers in K . Let $X = \text{Spec}(\mathcal{O}_K)$ and U be a nonempty open subset of X with complement S . Thus U consists of all but finitely many places of K . Let ℓ be a prime number that is invertible on U and let μ_ℓ be the sheaf of ℓ -th roots of unity. We are interested in the groups $H^i(U, \mu_\ell)$. To aid us in computing them and related cohomology groups, we have the *Poitou-Tate exact sequence* (see e.g. [MAD], Ch. I, §4, Theorem 4.10c):

$$\begin{aligned} 0 \rightarrow H^0(U, \mu_\ell) &\rightarrow \bigoplus_{v \in S} H^0(K_v, \mu_\ell) \rightarrow H^2(U, \mathbb{Z}/\ell\mathbb{Z})^* \rightarrow \\ H^1(U, \mu_\ell) &\rightarrow \bigoplus_{v \in S} H^1(K_v, \mu_\ell) \rightarrow H^1(U, \mathbb{Z}/\ell\mathbb{Z})^* \rightarrow \\ H^2(U, \mu_\ell) &\rightarrow \bigoplus_{v \in S} H^2(K_v, \mu_\ell) \rightarrow H^0(U, \mathbb{Z}/\ell\mathbb{Z})^* \rightarrow 0. \end{aligned}$$

This sequence summarizes many of the basic results from class field theory. Let K_S be a maximal extension of K that is unramified outside S and put $G_S = \text{Gal}(K_S/K)$. Then any sheaf \mathcal{F} on U may be regarded as a G_S -module, and we have $H^i(U, \mathcal{F}) \cong H^i(G_S, \mathcal{F})$. We shall often use this latter notation for the multiplicative group case. We are mainly interested in the middle line of the Poitou-Tate sequence:

$$(*)_{\mu_\ell} : H^1(G_S, \mu_\ell) \rightarrow \bigoplus_{v \in S} H^1(K_v, \mu_\ell) \rightarrow H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})^*$$

and the dual sequence obtained by taking the Pontryagin dual and using Tate local duality:

$$(*)_{\mathbb{Z}/\ell\mathbb{Z}} : H^1(G_S, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow \bigoplus_{v \in S} H^1(K_v, \mathbb{Z}/\ell\mathbb{Z}) \rightarrow H^1(G_S, \mu_\ell)^*.$$

For an elliptic curve E over K that has a smooth proper model \mathcal{E} over U on which ℓ is invertible, we have the *Cassels-Tate exact sequence* (see [MAD], Ch. II, §5, Theorem 5.6b):

$$(**) E(K)^{(\ell)} \rightarrow \bigoplus_{v \in S} E(K_v)^{(\ell)} \rightarrow H^1(U, \mathcal{E})\{\ell\}^* \rightarrow \text{III}(E)\{\ell\} \rightarrow 0.$$

Here (ℓ) denotes completion with respect to subgroups of ℓ -power index, $\{\ell\}$ denotes the ℓ -primary part of a torsion abelian group, and $\text{III}(E)$ is the Shafarevich-Tate group of everywhere locally trivial principal homogeneous spaces under E , which we assume to be finite.

We give here a very terse explanation of the common origin of these two exact sequences, as it is the key to our unified approach in the multiplicative group and elliptic curve cases. Let \mathcal{F} be a sheaf on U and $j_!\mathcal{F}$ denote extension of \mathcal{F} by zero from U to X . We denote by $H_c^i(U, \mathcal{F})$ the group $H^i(X, j_!\mathcal{F})$; this is *cohomology with compact support*. Then we have a long exact sequence of cohomology with support (see [MET], Chapter III, Proposition 1.25):

$$\cdots H_S^i(X, j_!\mathcal{F}) \rightarrow H^i(X, j_!\mathcal{F}) \rightarrow H^i(U, j^*j_!\mathcal{F}) \rightarrow H_S^{i+1}(X, j_!\mathcal{F}).$$

For a place v of K , let A_v^h denote the henselization of the local ring of X at v (one can also take the completion). Then using the identifications:

$$H_S^i(X, j_!\mathcal{F}) \cong \bigoplus_{v \in S} H_v^i(X, j_!\mathcal{F})$$

$$H_v^i(X, j_!\mathcal{F}) = H_v^i(A_v^h, j_!\mathcal{F})$$

$$H^i(K_v, \mathcal{F}) \cong H_v^{i+1}(A_v^h, j_!\mathcal{F})$$

for $v \in S$ (see [MAD], Proposition 1.1, page 182 for the last isomorphism, which uses the fact that we have a sheaf of the form $j_!\mathcal{F}$), we get the exact sequence

$$\cdots H_c^i(U, \mathcal{F}) \rightarrow H^i(U, \mathcal{F}) \rightarrow \bigoplus_{v \in S} H^i(K_v, \mathcal{F}) \rightarrow H_c^{i+1}(U, \mathcal{F}) \cdots$$

The Poitou-Tate and Cassels-Tate exact sequences are then derived from this one sequence by taking $\mathcal{F} = \mu_\ell$ (resp. $\mathcal{F} = \mathcal{E}$) and using the Artin-Verdier duality theorem (see e.g. [MAD], Chapter II, §3, Corollary 3.2) (resp. the duality theorem for abelian varieties (see [MAD], Chapter 3, §5, Theorem 5.2)).

3. Classical Index Calculus from the Perspective of Arithmetic Duality

Our approach to the discrete log problem for the multiplicative group of a finite field uses the Poitou-Tate exact sequence (*) in §2 above. For the discrete log problem for an elliptic curve \tilde{E} over a finite field with a point of order ℓ and a suitable lifting E of \tilde{E} to an algebraic number field K , we will use the Cassels-Tate sequence (**) in §2, where U is an open subset of $\text{Spec}(\mathcal{O}_K)$ on which E has good reduction and ℓ is invertible, and \mathcal{E} is a smooth proper model of E over U . In each

case, the method will be to find a suitable element of $H^1(U, \mathcal{F})$ of order ℓ against which to “test” a lifting to K of an element over the finite field whose discrete log we seek to compute and then use the reciprocity laws that are encoded in the exact sequences to create linear relations between the discrete logs.

We demonstrate below how the classical index calculus method emerges in this context as the result of one particular choice of testing Dirichlet character and method of lifting.

Let p and ℓ be odd primes such that $p \equiv 1 \pmod{\ell}$ but $p \not\equiv 1 \pmod{\ell^2}$. Given positive integers g and t such that $g \bmod p$ generates the group \mathbb{F}_p^* , we would like to compute $n \bmod \ell$ where $t = g^n$ in \mathbb{F}_p^* . We will fix g and denote the discrete-log t with respect to g as $\theta(t)$. The core of the classical index calculus method for solving the discrete-log problem in \mathbb{F}_p^* is to compute $\theta(q)$ for primes q up to a chosen bound B .

Let $K = \mathbb{Q}$, $X = \text{Spec}(\mathbb{Z})$, and $U = X - S$, where S is a finite set of primes containing ℓ . Consider the sequence $(*)_{\mathbb{Z}/\ell\mathbb{Z}}$ of the last section. The extension $\mathbb{Q}(\mu_p)/\mathbb{Q}$ is cyclic of degree $p - 1$. Since $p \equiv 1 \pmod{\ell}$, there is a unique sub-extension L/\mathbb{Q} of degree ℓ . We fix an isomorphism $\text{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/\ell\mathbb{Z}$ and denote by χ the corresponding Dirichlet character, which is ramified only at p . Then χ can be regarded as an element of $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ if $p \in S$. We have that $\mathbb{Z}_S^*/\mathbb{Z}_S^{*\ell} \cong H^1(G_S, \mu_\ell)$, and from $(*)_{\mathbb{Z}/\ell\mathbb{Z}}$ we have that for all $\alpha \in \mathbb{Z}_S^*$,

$$\sum_{v \in S} \langle \chi_v, \alpha_v \rangle = 0 \in \mathbb{Z}/\ell\mathbb{Z}.$$

Note that

$$\langle \chi_p, \alpha_p \rangle = \theta(\alpha) \langle \chi_p, g \rangle,$$

and for $q \in S - \{p\}$,

$$\langle \chi_q, \alpha_q \rangle = v_q(\alpha) \langle \chi_q, q \rangle,$$

where $v_q(\alpha)$ is the q -adic valuation of α .

Let F be the set of primes up to some bound B and let S be the set F together with p and ℓ . For $q \in F$, since $q \in \mathbb{Z}_S^*$ and q is a local unit at $v \neq q$ in S ,

$$0 = \sum_{v \in S} \langle \chi_v, q \rangle = \langle \chi_p, q \rangle + \langle \chi_q, q \rangle = \theta(q) \langle \chi_p, g \rangle + \langle \chi_q, q \rangle.$$

Hence,

$$\theta(q) = -(\langle \chi_p, g \rangle)^{-1} \langle \chi_q, q \rangle.$$

To compute $\theta(q)$ for all primes q in F , we generate random r so that $g^r \bmod p$ is B -smooth, that is

$$\alpha_r = g^r \bmod p = \prod_{q \in F} q^{e_q(r)}$$

with $e_q(r) \in \mathbb{Z}_{\geq 0}$. Since $\alpha_r \in \mathbb{Z}_S^*$, we have

$$0 = \sum_{v \in S} \langle \chi_v, (\alpha_r)_v \rangle = r \langle \chi_p, g \rangle + \sum_{q \in F} e_q(r) \langle \chi_q, q \rangle.$$

It follows that

$$\sum_{q \in F} e_q(r) \theta(q) = r.$$

With sufficiently many α_r that generate $\mathbb{Z}_F^*/\mathbb{Z}_F^{*\ell}$, we can solve for the unknown $\theta(q) \bmod \ell$. What we have derived is in essence the classical index calculus method.

We remark that similar reasoning as above shows that the image of $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ in $\bigoplus_{v \in S} H^1(\mathbb{Q}_v, \mathbb{Z}/\ell\mathbb{Z})$, where $S = F \cup \{p\}$, has \mathbb{F}_ℓ dimension one, and the classical index calculus method amounts to determining this image in a computationally efficient manner.

In the preceding discussion, we were able to explicitly construct the desired Dirichlet character because we were working with abelian extensions of \mathbb{Q} , about which we know enough to explicitly compute everything we need. In the discussion below we will be working with real quadratic fields, and there we know much less about how to explicitly construct abelian extensions. However, using the exact sequence $(*)_{\mu_\ell}$, we will demonstrate the existence of a suitable Dirichlet character by explicitly computing the \mathbb{F}_ℓ -dimensions of the first and second terms, and showing that the former is less than the latter. More generally, we use the following basic strategy to find a suitable testing element. In the multiplicative group case, look for an algebraic number field K such that the \mathbb{F}_ℓ -dimension of the first term of the middle row of $(*)_{\mu_\ell}$ is smaller than that of the second. This will then guarantee the existence of an element of order ℓ in $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})^*$. By lifting to units of a real quadratic field instead of to smooth integers in \mathbb{Z} , we are more able to compare and contrast the discrete log problems for the multiplicative group and for elliptic curves over finite fields. It is an artifact of class field theory that one can often demonstrate the existence of an abelian extension without there being an obvious way to construct it explicitly.

In the elliptic curve case, we look for an algebraic number field K together with an elliptic curve E/K that lifts \tilde{E} , such that $E(K)$ is of small rank, e.g. ≤ 2 . We also assume that at least one of the generators of the torsion-free quotient of $E(K)$ is not divisible by ℓ in $E(K_u)$ for all $u \in T$, where T consists of one place above p and both above ℓ in a quadratic extension K/\mathbb{Q} in which both p and ℓ split.

This approach will be developed in more detail in the next few sections.

4. Signature Calculus for the Multiplicative Group

4.1. Characters with Prescribed Ramification. Throughout this section, let p, ℓ be rational primes with $p \equiv 1 \pmod{\ell}$ and $\ell > 2$. Let K/\mathbb{Q} be a real quadratic extension where p and ℓ split. Let α be a fundamental unit of K . Let Σ be the set of all places over ℓ and p , together with all the archimedean places. For any place u of K let P_u denote the prime ideal corresponding to u . For any finite set S of places of K , let G_S denote the Galois group of a maximal extension of K that is unramified outside of S .

PROPOSITION 1. *Let S be a subset of Σ that contains both places over ℓ and both archimedean places. Suppose*

- (1) $\ell \nmid h_K$ where h_K is the class number of K ;
- (2) either $\alpha^{l-1} \not\equiv 1 \pmod{P_w^2}$ for some $w \in S$ over ℓ , or $\alpha^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{P_w}$ for some $w \in S$ over p (that is, locally α is not an ℓ -th power at either a place over ℓ or a place over p).

Then the \mathbb{F}_ℓ -dimension of $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ equals $n(S) - 1$ where $n(S)$ is the number of finite places in S .

Proof: Consider the sequence:

$$(*)_{\mu_\ell} : H^1(G_S, \mu_\ell) \xrightarrow{f} \bigoplus_{v \in S} H^1(K_v, \mu_\ell) \xrightarrow{\rho} H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})^* \rightarrow H^2(G_S, \mu_\ell) \xrightarrow{g} \bigoplus_{v \in S} H^2(K_v, \mu_\ell) \rightarrow \cdots$$

We claim that under the hypotheses of the proposition, ρ is surjective. To see this, the hypothesis that ℓ does not divide the class number of K implies that it does not divide the class number of \mathcal{O}_S . By Kummer theory, we then have that:

$$H^2(G_S, \mu_\ell) \cong Br(\mathcal{O}_S)[\ell].$$

But then the map g is injective, so ρ is surjective. Now consider the map

$$f : H^1(G_S, \mu_\ell) \xrightarrow{f} \bigoplus_{v \in S} H^1(K_v, \mu_\ell).$$

Again using the hypothesis that ℓ does not divide the class number of K , we have that:

$$\mathcal{O}_S^*/\mathcal{O}_S^{*\ell} \cong H^1(G_S, \mu_\ell).$$

Consider the exact sequence:

$$0 \rightarrow \mathcal{O}^* \rightarrow \mathcal{O}_S^* \rightarrow \mathbb{Z}S \rightarrow Cl(\mathcal{O}) \rightarrow Cl(\mathcal{O}_S) \rightarrow 0.$$

Going modulo ℓ and using the hypotheses of the theorem, we see that the sequence:

$$0 \rightarrow \mathcal{O}^*/\mathcal{O}^{*\ell} \rightarrow \mathcal{O}_S^*/\mathcal{O}_S^{*\ell} \rightarrow \mathbb{Z}S/\ell\mathbb{Z}S \rightarrow 0$$

is exact. This shows that the \mathbb{F}_ℓ -dimension of the group in the middle is $n(S) + 1$. The hypotheses about the units show that f is injective. The target has dimension $2n(S)$ because $H^1(K_v, \mu_\ell)$ is isomorphic to $\mathbb{Q}_v^*/\mathbb{Q}_v^{*\ell}$. If $v \mid p$, then this group is of dimension 2 over \mathbb{F}_ℓ because $\ell \mid p - 1$. If $v \nmid \ell$, then this group is also of dimension 2, spanned by a prime element of \mathbb{Q}_ℓ and by a 1-unit. Thus the cokernel of h is of dimension $n(S) - 1$. This completes the proof of the proposition.

PROPOSITION 2. *Let S be the set consisting of one place u over ℓ , one place v over p , and both archimedean places. Suppose*

- (1) $\ell \nmid h_K$ where h_K is the class number of K ;
- (2) $\alpha^{l-1} \not\equiv 1 \pmod{P_w^2}$ for all places $w \mid \ell$;
- (3) $\alpha^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{P_v}$.

Then the \mathbb{F}_ℓ -dimension of $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ is one. If χ is any nonzero element of this group, then χ is ramified at u and v .

Proof Suppose u, u' are the places over ℓ . Let R be the set consisting of u, u' and both archimedean places. Let T be the set consisting of u, u', v and both archimedean places. Then from Proposition 1 it follows that $H^1(G_R, \mathbb{Z}/\ell\mathbb{Z})$ has dimension one and $H^1(G_T, \mathbb{Z}/\ell\mathbb{Z})$ has dimension two. Hence there exists a nontrivial $\psi \in H^1(G_R, \mathbb{Z}/\ell\mathbb{Z})$, and some $\chi \in H^1(G_T, \mathbb{Z}/\ell\mathbb{Z}) - H^1(G_R, \mathbb{Z}/\ell\mathbb{Z})$. By construction χ is ramified at v , and by the condition on α at v we get $\langle \chi_v, \alpha_v \rangle \neq 0$. As for ψ , by the reciprocity law we have $\langle \psi_u, \alpha_u \rangle + \langle \psi_{u'}, \alpha_{u'} \rangle = 0$, so either $\langle \psi_u, \alpha_u \rangle$ and $\langle \psi_{u'}, \alpha_{u'} \rangle$ are both zero or both non-zero. But if both are zero then by the condition of α at u and u' it would follow that ψ is unramified at both places, violating the condition that ℓ does not divide the class number of K . Hence $\langle \psi_u, \alpha_u \rangle$ and $\langle \psi_{u'}, \alpha_{u'} \rangle$ are both non-zero. Since $\langle \psi_{u'}, \alpha_{u'} \rangle \neq 0$, there exists $c \in \mathbb{Z}/\ell\mathbb{Z}$ such that $\langle \chi_u, \alpha_{u'} \rangle = c \langle \psi_{u'}, \alpha_{u'} \rangle$, and letting $\phi = \chi - c\psi$, we have $\langle \phi_{u'}, \alpha_{u'} \rangle = 0$. Now $\phi \in H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ since $\langle \phi_{u'}, \alpha_{u'} \rangle = 0$, and ϕ is a nontrivial since $\langle \phi_v, \alpha_v \rangle = \langle \chi_v, \alpha_v \rangle \neq 0$. Hence $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ is of dimension at least one. Since ψ is ramified at u' , it follows that $\psi \notin H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$, and since $\psi \in H^1(G_R, \mathbb{Z}/\ell\mathbb{Z}) \subset H^1(G_T, \mathbb{Z}/\ell\mathbb{Z})$, it follows that $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ is a proper subset of $H^1(G_T, \mathbb{Z}/\ell\mathbb{Z})$, hence it can be of dimension at most one. We conclude that its dimension must be one, and the proposition follows.

Remarks. (i) We explain why we made the assumptions of Proposition 2, their necessity and sufficiency for the conclusion, and how they affect the signature computations later in the paper:

Condition (1) is made to ensure that the Dirichlet characters of degree ℓ that we get will not be everywhere unramified, as such characters would be of no use to us for the signature computation.

Conditions (2) and (3) are meant to ensure that there do not exist characters of K of degree ℓ that are ramified only at \mathfrak{p} or only at \mathfrak{l} . Such characters would not help our signature computation. For example, suppose the character χ is ramified at v and unramified everywhere else. Then if we pair χ with a global unit a of our real quadratic field, we would get that $\langle \chi_u, a_u \rangle = 0$ since χ is unramified at u and a is a unit. The reciprocity law would then give us that $\langle \chi_v, a_v \rangle = -\langle \chi_u, a_u \rangle = 0$, and this would not help us in the signature computation. If *neither* condition (2) nor (3) holds, then there are Dirichlet characters χ' and χ'' , one ramified only at u and the other ramified only at v . Thus, while the character $\chi = \chi' + \chi''$ is ramified at both u and v , this would not help for our signature computation, since for a global unit a , we would have:

$$\langle \chi_u, a_u \rangle = \langle \chi'_u, a_u \rangle + \langle \chi''_u, a_u \rangle = 0,$$

since χ' is ramified only at u and χ'' is unramified at u . Similarly for v .

(ii) One can give an alternative (and perhaps simpler) proof of Proposition 2 using the ideal theoretic formulation of class field theory. Briefly, using the hypotheses of the proposition, one easily calculates the ℓ -rank of the Galois group of the ray class field modulo $I = \mathfrak{p}\mathfrak{l}^2$, where \mathfrak{p} is an ideal of K lying over p and \mathfrak{l} is an ideal lying over ℓ . This is the maximal abelian extension of K with conductor

bounded by I , and its Galois group is isomorphic to a generalized class group by class field theory. Using basic exact sequences and the hypotheses of the proposition, we can explicitly calculate this class group. The reason why we did not write the proof this way is that we want to stress the analogy with elliptic curves, where the Poitou-Tate exact sequence has an analogue (the Cassels-Tate exact sequence), but the ideal theoretic formulation of class field theory has no known analogue.

Assuming the conditions in Proposition 2, then $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. Every nontrivial character in it is ramified at u and v and unramified at all other finite places; moreover, $\langle \chi_u, \alpha_u \rangle \neq 0$ and $\langle \chi_v, \alpha_v \rangle \neq 0$, and $\langle \chi_u, \alpha_u \rangle + \langle \chi_v, \alpha_v \rangle = 0$. This group of characters corresponds to a unique cyclic extension K_S of degree ℓ over K which is ramified at u and v and unramified at all other finite places.

At u , we take the class of $1 + \ell$ as the generator of the group $\mathcal{O}_{K_u}^* / \mathcal{O}_{K_u}^{*\ell} \cong \mathbb{Z}_\ell^* / \mathbb{Z}_\ell^{*\ell} \cong \mathbb{Z}/\ell\mathbb{Z}$. For $0 \neq \chi \in H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$, we call $\sigma_u(\chi) = \langle \chi_u, 1 + \ell_u \rangle$ the u -signature of χ .

Let $g \in \mathbb{Z}$ so that $g \bmod p$ generates the multiplicative group of \mathbb{F}_p . Then the class of g generates $\mathcal{O}_{K_v}^* / \mathcal{O}_{K_v}^{*\ell} \cong \mathbb{Z}_p^* / \mathbb{Z}_p^{*\ell} \cong \mathbb{Z}/\ell\mathbb{Z}$. For $\chi \in H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$, we call $\sigma_v(\chi) = \langle \chi_v, g_v \rangle \neq 0$ the v -signature of χ .

We call the pair $(\sigma_u(\chi), \sigma_v(\chi))$ the *signature* of χ . If we take χ' satisfying the conditions we have put, then $\chi' = a\chi$ for some $a \in (\mathbb{Z}/\ell\mathbb{Z})^*$, and hence we don't change $\sigma_u(\chi)\sigma_v(\chi)^{-1} \in \mathbb{Z}/\ell\mathbb{Z}$. This last quantity only depends on K_S and we call it the *ramification signature* of K_S ; it is nonzero.

4.2. DL and Signature Computation. In this section we show that the discrete logarithm problem in the multiplicative case is random polynomial time equivalent to computing the signature of cyclic extensions with prescribed ramification as described in Proposition 2.

DL Problem: Suppose we are given p, ℓ, g and a , where p and ℓ are prime with $p \equiv 1 \pmod{\ell}$, g is a generator for the group $\mathbb{F}_p^* \setminus \{\ell\}$ of elements killed by ℓ , and $a \in \mathbb{F}_p^* \setminus \{\ell\}$. Then compute $m \bmod \ell$ such that $a = g^m$ in \mathbb{F}_p .

Signature Computation Problem: Suppose we are given $K, p, \ell, u, u', v, \alpha$ and g , where $K = \mathbb{Q}(\sqrt{D})$ is a real quadratic field, ℓ, p are primes that split in K , u and u' are the two places of K over ℓ , v is a place of K over p , α is a unit of K , and g is a generator for \mathbb{F}_p^* such that: (1) the class number of K is not divisible by ℓ , (2) $\alpha^{l-1} \not\equiv 1 \pmod{P_u^2}$, $\alpha^{l-1} \not\equiv 1 \pmod{P_{u'}^2}$, and (3) $\alpha^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{P_v}$. Then compute the ramification signature, with respect to $1 + \ell$ and g , of the cyclic extension of degree ℓ over K which is ramified at u, v and unramified elsewhere.

THEOREM 1. *The problems DL and Signature Computation are random polynomial time equivalent.*

For the proof of the theorem, we first give a random polynomial time reduction from DL to Signature Computation. This part of the proof depends on some

heuristic assumption which will be made clear below.

Let $a = g^m$ in \mathbb{F}_p where m is to be computed. If $a^{\frac{p-1}{\ell}} = 1$, then $m \equiv 0 \pmod{\ell}$. So suppose $a^{\frac{p-1}{\ell}} \neq 1$. We lift a to some unit α of a real quadratic field K such that $\alpha \equiv a \pmod{v}$ for some place v of K over p . This can be done as follows.

- (1) Compute $b \in \mathbb{F}_p$ such that $ab = 1$ in \mathbb{F}_p .
- (2) $c \leftarrow 2^{-1}(a+b)$; $d \leftarrow 2^{-1}(a-b)$. Note that $c^2 - d^2 = 1$, and $a = c + d$. We may assume $d \neq 0$ otherwise $a^2 = 1$ and $m = (p-1)/2$ or $p-1$.
- (3) Treat d as an integer. Let $\gamma \in \bar{\mathbb{Q}}$ be such that $\gamma^2 = 1 + d^2$.
- (4) Check if $1 + d^2$ is a quadratic residue modulo ℓ . Otherwise substitute $d + rp$ for d for random r until the condition is met. This is to make sure that ℓ splits in K .
- (5) $\gamma^2 = 1 + d^2 \equiv c^2 \pmod{p}$ implies $\gamma \equiv c \pmod{v}$, and $\gamma \equiv -c \pmod{v'}$ where v and v' are the two places of K over p .
- (6) Let $\alpha = \gamma + d$. Then $\alpha \equiv c + d \equiv a \pmod{v}$. Note that the norm of α is $d^2 - \gamma^2 = -1$, so α is a unit of K .

We make the heuristic assumption that it is likely for K to satisfy the conditions in Proposition 2. (Note that condition (3) is satisfied since $\alpha \equiv a \pmod{v}$ and $a^{\frac{p-1}{\ell}} \neq 1$.) We argue below that computing the discrete logarithm m where $a = g^m$ is reduced to solving the Signature Computation problem on input K, p, ℓ, u, v, α and g , where $K = \mathbb{Q}(\gamma)$ with $\gamma^2 = 1 + d^2$, $\alpha = \gamma + d$, u and v are as constructed above. A simple analysis shows that the expected time complexity in constructing these objects is $O(\log^3 p)$.

For $\chi \in H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ that is ramified at u and v , and unramified elsewhere, we have

$$0 = \langle \chi_u, \alpha_u \rangle + \langle \chi_v, \alpha_v \rangle.$$

Moreover since $\alpha^{\frac{p-1}{\ell}} \not\equiv 1 \pmod{v}$, α generates $\mathcal{O}_{K_v}^*/\mathcal{O}_{K_v}^{*\ell}$, so $\langle \chi_v, \alpha_v \rangle \neq 0$, and it follows that $\langle \chi_u, \alpha_u \rangle \neq 0$.

In general for a field k and $a, b \in k^*$, we write $a \sim^l b$ if $a/b \in k^{*\ell}$.

We have $\alpha \sim^l g^m$ in K_v since $\alpha \equiv a \equiv g^m \pmod{v}$. Hence

$$\langle \chi_v, \alpha_v \rangle = \langle \chi_v, g_v^m \rangle = m \langle \chi_v, g_v \rangle.$$

Write $\alpha = \xi(1 + y\ell) \pmod{\ell^2}$ with $\xi^{\ell-1} = 1$ after identifying α with its isomorphic image in \mathbb{Q}_ℓ . Then $\alpha \sim^\ell (1 + \ell)^y$, and

$$0 = \langle \chi_u, \alpha_u \rangle = \langle \chi_u, (1 + \ell)_u^y \rangle = y \langle \chi_u, 1 + \ell_u \rangle.$$

Hence we have

$$\langle \chi_u, \alpha_u \rangle + \langle \chi_v, \alpha_v \rangle = y \langle \chi_u, 1 + \ell_u \rangle + m \langle \chi_v, g_v \rangle.$$

So $y\sigma_u(\chi) + m\sigma_v(\chi) = 0$. From this we see that if the ramification signature $\sigma_u(\chi)(\sigma_v(\chi))^{-1}$ is determined then m is determined. The expected time in this

reduction is $O(\log^3 p)$.

Next we give a random polynomial time reduction from Signature Computation on input K, p, ℓ, u, v, α and g , to DL on input p, ℓ, g and a where $\alpha \equiv a \pmod{v}$.

Call the oracle to DL on input p, ℓ, g and a to compute m such that $g^m = a \pmod{p}$. Then $\alpha \equiv g^m \pmod{v}$.

Write $\alpha = \xi(1 + y\ell) \pmod{\ell^2}$ with $\xi^{\ell-1} = 1$ after identifying α with its isomorphic image in \mathbb{Q}_ℓ . Then $\alpha \sim^\ell (1 + \ell)^y$. Again, $\xi \pmod{\ell^2}$ and hence y can be computed efficiently in time $O(|\alpha| \log \ell + \log^3 \ell) = O(|\alpha| \log p + \log^3 p)$.

For $\chi \in H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ that is ramified at u and v , and unramified elsewhere, we have as before $\langle \chi_v, \alpha_v \rangle = \langle \chi_v, g_v^m \rangle = m \langle \chi_v, g_v \rangle$, and $\langle \chi_u, \alpha_u \rangle = \langle \chi_u, (1 + \ell)_u^y \rangle = y \langle \chi_u, 1 + \ell_u \rangle$. Hence

$$0 = \langle \chi_u, \alpha_u \rangle + \langle \chi_v, \alpha_v \rangle = y \langle \chi_u, 1 + \ell_u \rangle + m \langle \chi_v, g_v \rangle$$

from this we can determine the signature $\sigma_u(\chi)(\sigma_v(\chi))^{-1}$. The expected running time in this reduction is $O(|\alpha| \log p + \log^3 p)$

5. Signature Calculus for ECDL

5.1. Preliminaries. In this section we will demonstrate the existence of principal homogeneous spaces of order ℓ under elliptic curves over number fields with prescribed ramification. We begin by describing $H^1(K_v, E)[\ell]$ in general terms when E has good reduction at v .

LEMMA 1. *Let K_v be a local field with finite residue field k . Let E be an elliptic curve defined over K_v with good reduction.*

- (1) *Suppose the characteristic of k is ℓ . Then $H^1(K_v, E)[\ell] \cong \mathbb{Z}/\ell\mathbb{Z}$ if $K_v \cong \mathbb{Q}_\ell$ and $\ell \nmid \#\tilde{E}(k)$.*
- (2) *Suppose the characteristic of k is not ℓ . Then*
 - (a) $H^1(K_v, E)[\ell] = 0$ *if $\ell \nmid \#\tilde{E}(k)$;*
 - (b) $H^1(K_v, E)[\ell] \cong \mathbb{Z}/\ell\mathbb{Z}$ *if $\ell \mid \#\tilde{E}(k)$ but $\ell^2 \nmid \#\tilde{E}(k)$.*

Proof

Let $E_1(K_v)$ be the kernel of the reduction map from $E(K_v)$ to $\tilde{E}(k)$. From the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & E_1(K_v) & \rightarrow & E(K_v) & \rightarrow & \tilde{E}(k) \rightarrow 0 \\ & & \downarrow \ell & & \downarrow \ell & & \downarrow \ell \\ 0 & \rightarrow & E_1(K_v) & \rightarrow & E(K_v) & \rightarrow & \tilde{E}(k) \rightarrow 0 \end{array}$$

and the snake lemma, we get the exact sequence

$$\begin{aligned} 0 \rightarrow E_1(K_v)[\ell] \rightarrow E(K_v)[\ell] \rightarrow \tilde{E}(k)[\ell] \rightarrow E_1(K_v)/\ell E_1(K_v) \\ \rightarrow E(K_v)/\ell E(K_v) \rightarrow \tilde{E}(k)/\ell \tilde{E}(k) \rightarrow 0. \end{aligned}$$

If ℓ does not divide the order of $\tilde{E}(k)$, then $\tilde{E}(k)[\ell]$ and $\tilde{E}(k)/\ell\tilde{E}(k)$ are both 0. Hence $E(K_v)/\ell E(K_v) \cong E_1(K_v)/\ell E_1(K_v)$.

To prove (1) suppose the characteristic of k is ℓ . If $K_v \cong \mathbb{Q}_\ell$, then $E_1(K_v)/\ell E_1(K_v) \cong \mathbb{Z}/\ell\mathbb{Z}$. If moreover $|\tilde{E}(k)|$ is not divisible by ℓ , then $E(K_v)/\ell E(K_v) \cong E_1(K_v)/\ell E_1(K_v) \cong \mathbb{Z}/\ell\mathbb{Z}$, hence $H^1(K_v, E)[\ell] \cong \mathbb{Z}/\ell\mathbb{Z}$ by local duality.

To prove (2), suppose the characteristic of k is not ℓ . Then $E_1(K_v)/\ell E_1(K_v) = 0$, and it follows from the long exact sequence that $E(K_v)/\ell E(K_v) \cong \tilde{E}(k)/\ell\tilde{E}(k)$. If ℓ does not divide the order of $\tilde{E}(k)$, then $E(K_v)/\ell E(K_v) \cong \tilde{E}(k)/\ell\tilde{E}(k) = 0$, and by local duality, $H^1(K_v, E)[\ell] = 0$. This proves 2(a). If $|\tilde{E}(k)|$ is divisible by ℓ but not ℓ^2 , then $\tilde{E}(k)/\ell\tilde{E}(k) \cong \mathbb{Z}/\ell\mathbb{Z}$. Hence $E(K_v)/\ell E(K_v) \cong \tilde{E}(k)/\ell\tilde{E}(k) \cong \mathbb{Z}/\ell\mathbb{Z}$, and by local duality, $H^1(K_v, E)[\ell] = \mathbb{Z}/\ell\mathbb{Z}$. Thus 2(b) is proved.

5.1.1. Ranks of Quadratic Twists of Elliptic Curves over \mathbb{Q} . Let E be an elliptic curve over \mathbb{Q} and fix a Weierstrass equation for E :

$$y^2 = x^3 + ax + b.$$

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension of \mathbb{Q} and let E_d be the quadratic twist of E given by the equation

$$dy^2 = x^3 + ax + b.$$

Let G be the Galois group of K over \mathbb{Q} and σ a generator of G . Denote by V the group $E(K) \otimes_{\mathbb{Z}} \mathbb{Q}$, by V^+ the fixed space by σ , and by V^- the subspace of V where σ acts by -1 . Now

$$V = V^+ \oplus V^-,$$

$V^+ = E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and we see easily that $V^- = E_d(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$, via the isomorphism sending a point (x, y) in $E_d(\mathbb{Q})$ to $(x, \sqrt{d}y)$ in V^- .

In the algorithm in § 5.3 below, it will help to have a lifting E/\mathbb{Q} of our original elliptic curve \tilde{E}/\mathbb{F}_p such that both $E(\mathbb{Q})$ and $E_d(\mathbb{Q})$ are of rank one. Standard conjectures about the behavior of the rank of the Mordell-Weil group of an elliptic curve predict that it should be quite possible to find such a situation. For example, a conjecture of Goldfeld [G] says that the rank of a quadratic twist E_d of an elliptic curve E over \mathbb{Q} should be on average as small as the sign of the functional equation of its L -function would allow, i.e. either 0 or 1, depending on whether this sign is $+1$ or -1 . In fact, assuming the Riemann hypothesis for all of the curves E_d , Heath-Brown [HB] has proved that at least $1/4$ of all the E_d with the sign in the functional equation of the L -function being $+1$ will have rank 0 and at least $3/4$ of all the E_d with the sign being -1 will have rank 1 (see [H-B], Theorem 4). In the algorithm, we will first lift \tilde{E}/\mathbb{F}_p to E/\mathbb{Q} that has rank at least one by construction, and we will make the heuristic assumption that $E(\mathbb{Q})$ is of rank exactly one and therefore the sign of the functional equation is -1 (see [BMSW], §3 for why this is considered to be reasonable). Using ([RS], Theorem 7.2), Heath-Brown's result just mentioned, and taking sufficiently many random d , we can heuristically arrange for the sign

of the functional equation of E_d to be equal to -1 and for $E_d(\mathbb{Q})$ to have rank 1. When we make the heuristic assumption in § 5.3 below that the rank of $E(K)$ is exactly two, we shall mean this.

5.1.2. The Group $E(K_v)/\ell$ at Bad Reduction Primes v of E . Let \tilde{E} be an elliptic curve over \mathbb{F}_p , $p \geq 5$, given in Weierstrass form by an affine equation

$$y^2 = x^3 + \tilde{a}x + \tilde{b}.$$

In the algorithm below, we will want to lift \tilde{E} to an elliptic curve E over \mathbb{Q} with Weierstrass equation

$$y^2 = x^3 + ax + b,$$

having good reduction at ℓ and such that at primes v of bad reduction, $E(K_v)/\ell = 0$. We give a heuristic here about why this should be possible. In our lifting in the algorithm presented in § 5.3, $|a|$ is at most p^2 and $|b|$ is at most p^4 , so the discriminant Δ of a minimal Weierstrass equation for E is of order at most p^8 . At a prime v of split multiplicative reduction prime, the group of connected components of the Néron model of E over the ring of integers of \mathbb{Q}_v will be of order the power of v in the discriminant. Since ℓ is of the same order as p , this power is very unlikely to be divisible by ℓ . At other primes of bad reduction, the group of connected components is of order at most 4 (see [Si1], Ch. VII, Theorem 6.1). Thus the order of the group of components is unlikely to be divisible by ℓ . We claim that this implies that for any bad reduction place v of E , $E(K_v)/\ell = 0$. To see this, recall that $E(K_v)$ has a filtration:

$$E(K_v) \supseteq E_0(K_v) \supseteq E_1(K_v),$$

where $E_0(K_v)$ is the group of points specializing to points of the smooth locus \tilde{E}'^0 of the special fibre \tilde{E}' of the minimal regular proper model \mathcal{E} of E over the ring of integers R of K_v and $E_1(K_v)$ is the kernel of the reduction map

$$E_0(K_v) \rightarrow \tilde{E}'^0(\mathbb{F}).$$

Now $E(K_v)/E_0(K_v)$ is the group of connected components of the special fibre of the Néron model of E over R , and $E_1(K_v)$ is a pro- v -group, where v is the residue characteristic of K_v . $E_0(K_v)/E_1(K_v)$ is the group of points on the connected component of identity of the special fibre of the Néron model. This last group is isomorphic to either the additive group or the multiplicative group of the residue field, \mathbb{F}_v . Because E has good reduction at ℓ , $v \neq \ell$, ℓ is large and v is relatively small compared to ℓ , it is unlikely that ℓ will divide $v - 1$. Thus it is likely that $E(K_v)/\ell = 0$ and we shall use this heuristic in the algorithm in § 5.3 below.

5.2. Principal Homogeneous Spaces Ramified over p and ℓ . Throughout this section, let p, ℓ be odd, rational primes. Let K/\mathbb{Q} be a real quadratic extension, $X = \text{Spec}(\mathcal{O}_K)$ and Σ be the set of all places at which E has bad reduction, together with all the archimedean places. Let \mathcal{E} be a smooth proper model of E over the open subset $U = X - \Sigma$. If S is any set of places of K containing Σ , denote by U_S the open set $X - S$. We denote by $\text{III}(E)$ the Shafarevich-Tate group of everywhere locally trivial principal homogeneous spaces under E over K .

PROPOSITION 3. *Let S be a finite set of places of K containing all bad reduction places of E and the places above ℓ . Then if $\text{III}(E)\{\ell\} = 0$, we have the exact sequence:*

$$E(K)/\ell \rightarrow \prod_{v \in S} E(K_v)/\ell \rightarrow (H^1(U_S, \mathcal{E})[\ell])^* \rightarrow 0.$$

Proof: Consider the Cassels-Tate exact sequence

$$(**) E(K)^{(\ell)} \rightarrow \prod_{v \in S} E(K_v)^{(\ell)} \rightarrow H^1(U_S, \mathcal{E})\{\ell\}^* \rightarrow \text{III}(E)\{\ell\} \rightarrow 0.$$

LEMMA 2. *Let B be a torsion abelian group such that $B[\ell^n]$ and $B/\ell^n B$ are finite groups. Then we have*

$$B[\ell]^* \cong B^*/\ell B^*$$

and

$$B\{\ell\}^* \cong B^{*(\ell)}$$

Proof:

Let n be a positive integer and consider the tautological exact sequence:

$$0 \rightarrow B[\ell^n] \rightarrow B \xrightarrow{\ell^n} B \rightarrow B/\ell^n B \rightarrow 0.$$

Since the functor $*$ (see §1 for notation) is exact on the category of locally compact abelian groups, we get the exact sequence:

$$0 \rightarrow (B/\ell^n B)^* \rightarrow B^* \xrightarrow{\ell^n} B^* \rightarrow B[\ell^n]^* \rightarrow 0.$$

We then get the first conclusion of the lemma by taking $n = 1$ and the second by passing to the inverse limit over n . This completes the proof of the lemma.

The proposition then follows from the lemma, the assumption that $\text{III}(E)\{\ell\} = 0$, and the Cassels-Tate sequence above by reducing the terms mod ℓ .

For the remainder of this section we assume that p and ℓ split in K , and that E has good reduction at p and ℓ , with $\#\tilde{E}(\mathbb{F}_p) = \ell$ and $\ell \neq \#\tilde{E}(\mathbb{F}_\ell)$. Moreover, because we assume that ℓ is sufficiently large, a theorem of Kamienny [Ka] ensures that $E(L)[\ell]$ is trivial for all quadratic extensions L over \mathbb{Q} . Finally, we assume that $E(K_v)/\ell = 0$ for all bad reduction places v of E (see §1.3 for why this is reasonable, heuristically).

PROPOSITION 4. *Let S be a finite set of places of K containing all bad reduction places of E and the places above ℓ . S may or may not contain places above p , but assume that it contains no good reduction places that do not divide ℓ or p . Suppose*

- (1) $\text{III}(E)\{\ell\} = 0$;
- (2) *the map $E(K)/\ell \rightarrow E(K_u)/\ell \oplus E(K_{u'})/\ell$ is an isomorphism, where u and u' are the two places of K over ℓ .*

Then the \mathbb{F}_ℓ -dimension of $H^1(U, \mathcal{E})[\ell]$ equals $n(S) - 2$ where $n(S)$ is the number of finite places in $S - \Sigma$.

Proof: Since $\text{III}(E)\{\ell\} = 0$, we have the exact sequence

$$E(K)/\ell \rightarrow \prod_{v \in S} E(K_v)/\ell \rightarrow (H^1(U_S, \mathcal{E})[\ell])^* \rightarrow 0$$

by Proposition 3. The middle group in the sequence $\prod_{v \in S} E(K_v)/\ell$ is isomorphic to the direct sum of $n(S)$ copies of $\mathbb{Z}/\ell\mathbb{Z}$ by Lemma 1. Since the map

$$E(K)/\ell \rightarrow E(K_u)/\ell \oplus E(K_{u'})/\ell \cong \mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/\ell\mathbb{Z}$$

is an isomorphism, it follows that the image of the map

$$E(K)/\ell \rightarrow \prod_{v \in S} E(K_v)/\ell$$

is isomorphic to $\mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/\ell\mathbb{Z}$. Hence the \mathbb{F}_ℓ -dimension of $H^1(U_S, \mathcal{E})[\ell]$ equals $n(S) - 2$.

PROPOSITION 5. *Let S be the set consisting of all bad reduction places of E , together with the two places u and u' over ℓ , and one place v over p . Suppose*

- (1) $\text{III}(E)\{\ell\} = 0$;
- (2) *the map $E(K)/\ell \rightarrow E(K_u)/\ell \oplus E(K_{u'})/\ell$ is an isomorphism.*

Then the \mathbb{F}_ℓ -dimension of $H^1(U, \mathcal{E})[\ell]$ is one. Moreover, every nontrivial element of $H^1(U, \mathcal{E})[\ell]$ is ramified at v .

Proof Suppose u, u' are the places over ℓ , v, v' the places over p . Let $R = \Sigma \cup \{u, u'\}$ and $T = \Sigma \cup \{u, u', v\}$. Then from Proposition 4 we know that $H^1(U_R, \mathcal{E})[\ell]$ has dimension zero and $H^1(U_T, \mathcal{E})[\ell]$ has dimension one. So there exists $\chi \in H^1(U_T, \mathcal{E})[\ell] - H^1(U_R, \mathcal{E})[\ell]$ and χ must be ramified at v . This completes the proof of the proposition.

We remark that in the proposition above the assumption that the map $E(K)/\ell \rightarrow E(K_u)/\ell \oplus E(K_{u'})/\ell$ is an isomorphism can be replaced by the assumption that the image of $E(K)/\ell$ in $E(K_u)/\ell \oplus E(K_{u'})/\ell$ and in $E(K_v)/\ell \oplus E(K_u)/\ell \oplus E(K_{u'})/\ell$ are both of \mathbb{F}_ℓ -dimension two.

For $w = u, u', v$, let $R_w \in E(K_w) - \ell E(K_w)$, so that the class of R_w generates $E(K_w)/\ell$. For $\chi \in H^1(U_S, \mathcal{E})[\ell]$ and w a place of K , we call $a_w = \langle \chi_w, R_w \rangle$ the w -signature of χ . We call $(a_u, a_{u'}, a_v)$ the signature of χ with respect to $R_u, R_{u'}$ and R_v . Proposition 5 implies that $\langle \chi_v, R_v \rangle \neq 0$ for any nontrivial $\chi \in H^1(U_S, \mathcal{E})[\ell]$, and $(\frac{\langle \chi_u, R_u \rangle}{\langle \chi_v, R_v \rangle}, \frac{\langle \chi_{u'}, R_{u'} \rangle}{\langle \chi_v, R_v \rangle})$ is the same for all such χ . We call this pair the *signature* of $H^1(U, \mathcal{E})[\ell]$ with respect to $R_u, R_{u'}$ and R_v .

Since the pairing between $H^1(K_v, E)[\ell]$ and $E(K_v)/\ell E(K_v)$ is perfect, both being isomorphic to $\mathbb{Z}_\ell/\mathbb{Z}_\ell$, there is a unique $\psi_v \in H^1(K_v, E)[\ell]$ such that $\langle \psi_v, R_v \rangle = 1$. Similarly, there is a unique $\psi_u \in H^1(K_u, E)[\ell]$ such that $\langle \psi_u, R_u \rangle = 1$, and a unique $\psi_{u'} \in H^1(K_{u'}, E)[\ell]$ such that $\langle \psi_{u'}, R_{u'} \rangle = 1$. Let $\chi \in H^1(U_S, \mathcal{E})[\ell]$. Suppose $\chi_v = a_v \psi_v$, $\chi_u = a_u \psi_u$ and $\chi_{u'} = a_{u'} \psi_{u'}$. Then $\langle \chi_v, R_v \rangle = a_v$, $\langle \chi_u, R_u \rangle = a_u$ and $\langle \chi_{u'}, R_{u'} \rangle = a_{u'}$. So $a_u, a_{u'}$ and a_v constitute the signature for χ with respect to $R_u, R_{u'}$ and R_v . Thus the signature $(a_u, a_{u'}, a_v)$ succinctly represents the localization of χ at the ramified places. These localizations in turn determine χ uniquely, since the Shafarevich-Tate group

is assumed to have trivial ℓ -part. Therefore, the signature of χ can be regarded as a succinct representation of χ (by determining its localization at u, u' and v as $\chi_u = a_u\psi_u$, $\chi_{u'} = a_{u'}\psi_{u'}$, and $\chi_v = a_v\psi_v$). We note that this representation requires only $O(\log \ell)$ bits whereas an explicit description of χ may require $\Omega(\ell)$ bits.

Suppose in addition to the map $E(K)/\ell \rightarrow E(K_u)/\ell \oplus E(K_{u'})/\ell$ being an isomorphism, we assume that the map $E(K)/\ell \rightarrow E(K_v)/\ell$ is nontrivial. In this case we may obtain R_w 's as follows. Let $Q, R \in E(K)$ so that their classes generate $E(K)/\ell$. Suppose without loss of generality that the class of Q is nontrivial in $E(K_u)/\ell$ and the class of R is nontrivial in $E(K_{u'})/\ell$. As $E(K)/\ell \rightarrow E(K_v)/\ell$ is nontrivial, the class of either Q or R is nontrivial in $E(K_v)/\ell$. Suppose without loss of generality the class of Q is nontrivial in $E(K_v)/\ell$. Then we may take $R_v = Q$, $R_u = Q$ and $R_{u'} = R$.

5.3. ECDL and Signature Computation. In this section we show that the elliptic curve discrete logarithm problem is random polynomial time equivalent to computing the signature of homogeneous spaces with prescribed ramification as described in Proposition 5.

ECDL: Given $p, \ell, \tilde{E}, \tilde{Q}$ and \tilde{R} , where p and ℓ are prime, \tilde{E} is an elliptic curve defined over \mathbb{F}_p with $\#\tilde{E}(\mathbb{F}_p) = \ell$, and non-zero points $\tilde{Q}, \tilde{R} \in \tilde{E}(\mathbb{F}_p)$, to determine m so that $\tilde{R} = m\tilde{Q}$.

Homogeneous Space Signature Computation: Suppose we are given p, ℓ, K, E, v, Q, R , where p and ℓ are prime, K is a quadratic field where p and ℓ both split, E is an elliptic curve defined over K with $\text{III}(E)\{\ell\} = 0$ and the discriminant of E being prime to ℓ , v is a place of K over p , Q and $R \in E(K)$ such that $Q \not\equiv 0 \pmod{\ell E(K_v)}$ and the images of R and Q in $E(K_u)/\ell \oplus E(K_{u'})/\ell$ are independent, where u and u' are the two places of K over ℓ . Then compute the signature of $H^1(U_S, \mathcal{E})[\ell]$ with respect to $\rho_v = Q$, $\rho_u = Q$ and $\rho_{u'} = R$, where S is the set consisting of u, u', v and all places of bad reduction of E . (Note that ρ_w generates $E(K_w)/\ell E(K_w)$ for $w = u, u', v$.)

THEOREM 2. *The problems ECDL and Homogeneous Space Signature Computation are random polynomial time equivalent.*

For the proof of the theorem, we first give a random polynomial time reduction from ECDL to Homogeneous Space Signature Computation. This part of the proof depends on some heuristic assumptions which will be made clear below.

Given \tilde{E}/\mathbb{F}_p where $\tilde{E}(\mathbb{F}_p)[\ell] = \langle \tilde{Q} \rangle$, and \tilde{R} , we are to compute m so that $\tilde{R} = m\tilde{Q}$. Steps 1-3 of the reduction construct an instance p, ℓ, K, E, v, Q, R of the Homogeneous Space Signature Computation problem.

1. Construct E/\mathbb{Q} with $Q \in E(\mathbb{Q})$ such that $\tilde{Q} = Q \pmod{p}$. This can be done as follows. Suppose \tilde{E} is specified by an affine equation $y^2 = x^3 + \bar{a}x + \bar{b}$ where

$\bar{a} = a \bmod p$, $\bar{b} = b \bmod p$ with $0 \leq a, b < p$ and $\tilde{Q} = (u \bmod p, v \bmod p)$ with $0 < u, v < p$. Choose a random integer r , $0 \leq r < p$, and let $Q = (u, v + rp)$. Let $b_r = (v + rp)^2 - (u^3 + au)$. Then $Q \in E_r(\mathbb{Q})$ where E_r is the elliptic curve with the affine equation $y^2 = x^3 + ax + b_r$. Set $E = E_r$. The point Q cannot be torsion for otherwise it would have to be in $E(\mathbb{Q})[\ell]$, which has no non-zero point since ℓ is big. The height of Q is far smaller than that of a point in $\ell E(\mathbb{Q})$, so Q is not in $\ell E(\mathbb{Q})$. Since $\tilde{E}(\mathbb{F}_p)[\ell] \cong \mathbb{Z}/\ell\mathbb{Z}$, $E(\mathbb{Q}_p)/\ell \cong \tilde{E}(\mathbb{F}_p)/\ell \cong \mathbb{Z}/\ell\mathbb{Z}$ and the class of Q generates $E(\mathbb{Q}_p)/\ell$.

2. Check that E has good reduction at ℓ and that $|\tilde{E}(\mathbb{F}_\ell)|$ is not divisible by ℓ . Otherwise, go back to 1. to find a different E .

3. Lift \tilde{R} to $R \in E(K)$ where K/\mathbb{Q} is a quadratic extension in which p and ℓ both split. This can be done as follows. Suppose E is defined by the affine equation $y^2 = x^3 + ax + c$. Suppose $\tilde{R} = (\mu \bmod p, \nu \bmod p)$ with $0 < \mu, \nu < p$. Choose a random positive integer $r < p$. Set $\mu_r = \mu + rp$. Let β be a root of $y^2 = \mu_r^3 + a\mu_r + c$. Then (μ_r, β) is a lift of \tilde{R} in $E(K)$ where $K = \mathbb{Q}(\beta)$. By construction p splits in K ,

$$E(K_v)/\ell \cong E(\mathbb{Q}_p)/\ell \cong \tilde{E}(\mathbb{F}_p)/\ell \cong \mathbb{Z}/\ell\mathbb{Z}$$

and $R - mQ \in \ell E(K_v)$. Check that ℓ splits in K and that the images of R and Q in $E(K_u)/\ell \oplus E(K_{u'})/\ell$ are independent; otherwise repeat the above steps with a different r until a suitable K is found. Say the class of Q is nontrivial in $E(K_u)/\ell$ and the class of R is nontrivial in $E(K_{u'})/\ell$.

4. Call the oracle for the Homogeneous Space Signature Computation on input p, ℓ, E, K, Q, R, v to compute the signature (α, β) of $H^1(U_S, \mathcal{E})[\ell]$ with respect to $\rho_v = Q$, $\rho_u = Q$ and $\rho_{u'} = R$ (where S is the set consisting of u, u', v and all places of bad reduction of E). Then for all nontrivial $\chi \in H^1(U_S, \mathcal{E})[\ell]$, $\alpha = \frac{\langle \chi_u, Q_u \rangle}{\langle \chi_v, Q_v \rangle}$ and $\beta = \frac{\langle \chi_{u'}, R_{u'} \rangle}{\langle \chi_v, Q_v \rangle}$.

5. Identify K_u with \mathbb{Q}_ℓ and compute n so that $R \equiv nQ \pmod{\ell E(K_u)}$ as follows. Compute $d = |\tilde{E}(\mathbb{F}_\ell)|$. Observe that dQ and dR are both in $E_1(\mathbb{Q}_\ell)$. Compute n such that $n(dQ) \equiv (dR) \pmod{\ell}$ in $E_1(\mathbb{Q}_\ell)$. Then $d(nQ - R) = \ell Z$ for some $Z \in E_1(\mathbb{Q}_\ell)$. Since d is not divisible by ℓ , $d^{-1} \in \mathbb{Z}_\ell$, so $nQ - R = d^{-1}\ell Z = \ell(d^{-1}Z) \in \ell E(\mathbb{Q}_\ell)$.

6. Now

$$\begin{aligned} 0 &= \sum_{w \in \{v, u, u'\}} \langle \chi_w, R_w \rangle \\ &= m \langle \chi_v, Q_v \rangle + n \langle \chi_u, Q_u \rangle + \langle \chi_{u'}, R_{u'} \rangle. \end{aligned}$$

From this we get $m + n\alpha + \beta \equiv 0 \pmod{\ell}$. Hence m can be determined.

We make the heuristic assumption that it is likely for E and K to satisfy the conditions in Proposition 4. Note that by construction $E(\mathbb{Q})$ is of rank at least one. The points Q and R are likely to be integrally independent in $E(K)$ as they both have small height by construction. So $E(K)$ is likely to be of rank at least two and

we make the heuristic assumption that with nontrivial probability its rank is exactly two. Moreover, since $Q \in E(\mathbb{Q})$ and $R \in E(K) - E(\mathbb{Q})$, the images of Q and R are likely to be independent in $E(K_u)/\ell \oplus E(K_{u'})/\ell$, heuristically speaking. The expected running time of this reduction is dominated by Step 2 where the number of rational points on the reduction of $E \bmod \ell$ is counted. The running time of that step is $O(\log^8 \ell)$ [Sc], hence it is $O(\log^8 p)$.

Next we give a random polynomial time reduction from Homogeneous Space Signature Computation with input p, ℓ, E, K, Q, R, v to ECDL with input $p, \ell, \tilde{E}, \tilde{Q}, \tilde{R}$, where \tilde{E} is the reduction of $E \bmod v$, \tilde{Q} (resp. \tilde{R}) is the reduction of Q (resp. R) mod v .

For any nontrivial $\chi \in H^1(K, E)[\ell]$ that is unramified away from u, u' and v , we have

$$\begin{aligned} \langle \chi_v, Q_v \rangle + \langle \chi_u, Q_u \rangle + \langle \chi_{u'}, Q_{u'} \rangle &= 0, \\ \langle \chi_v, R_v \rangle + \langle \chi_u, R_u \rangle + \langle \chi_{u'}, R_{u'} \rangle &= 0. \end{aligned}$$

Suppose $Q = a_w \rho_w \pmod{\ell E(K_w)}$ and $R = b_w \rho_w \pmod{\ell E(K_w)}$ for $w = u, u', v$. Note that from Lemma 1, a_v and b_v can be computed by solving ECDL on the reduction of E modulo v . On the other hand a_w, b_w for $w = u, u'$, can be computed in a manner as described in Step 5 above.

Then we get

$$\begin{aligned} a_v \langle \chi_v, \rho_v \rangle + a_u \langle \chi_u, \rho_u \rangle + a_{u'} \langle \chi_{u'}, \rho_{u'} \rangle &= 0, \\ b_v \langle \chi_v, \rho_v \rangle + b_u \langle \chi_u, \rho_u \rangle + b_{u'} \langle \chi_{u'}, \rho_{u'} \rangle &= 0 \end{aligned}$$

Condition (2) of Proposition 4 implies that the two relations above are linearly independent. From these we can compute the signature of χ ; that is $(\frac{\langle \chi_u, \rho_u \rangle}{\langle \chi_v, \rho_v \rangle}, \frac{\langle \chi_{u'}, \rho_{u'} \rangle}{\langle \chi_v, \rho_v \rangle})$. The expected running time of this reduction can be shown to be $O(\log^4 p) + O(M \log p)$ where M is the maximum of the lengths of R, Q and D .

6. Feasibility of Index Calculus

We will derive an index calculus method for the signature computation problem of Dirichlet characters. We will discuss why a similar method cannot work for principal homogeneous spaces.

6.1. Index Calculus for Signature Computation of Dirichlet Characters.

Suppose we are given a real quadratic field K , primes ℓ, p , places u, v satisfying the conditions in Proposition 2. Let $K = \mathbb{Q}(\alpha)$ with $\alpha^2 \in \mathbb{Z}_{>0}$. To compute the signature of $\chi \in H^1(K, \mathbb{Z}/\ell\mathbb{Z})$ that is ramified precisely at u and v , we generate random algebraic integers $\beta = r\alpha + s$ with $r, s \in \mathbb{Z}$ so that $r\alpha + s \equiv g \pmod{v}$ and $\beta \sim (1 + \ell)^a$ at u for some a . Now suppose the norm of β is B -smooth for some integer B . Then

$$0 = \sum_w \langle \chi_w, \beta_w \rangle = \langle \chi_v, g_v \rangle + a \langle \chi_u, 1 + \ell_u \rangle + \sum_w e_w \langle \chi_w, \pi_w \rangle,$$

where w in the last sum ranges over all places of K of norm less than B , π_w is a local parameter at w , and e_w is the valuation of β under w . Hence we have obtained

a $\mathbb{Z}/\ell\mathbb{Z}$ -linear relation on $(\langle \chi_v, g_v \rangle)^{-1} \langle \chi_u, 1 + \ell_u \rangle$, and $(\langle \chi_v, g_v \rangle)^{-1} \langle \chi_w, \pi_w \rangle$. With $O(B)$ relations we can solve for all these unknowns, in particular the signature $(\langle \chi_v, g_v \rangle)^{-1} \langle \chi_u, 1 + \ell_u \rangle$.

6.2. The Elliptic Curve Case. We see that one important reason why index calculus is viable in the multiplicative case is due to the fact that locally unramified Dirichlet characters can be paired nontrivially with non-units. For the elliptic curve case, pairing a principal homogeneous space χ and a global point α yields similarly a relation:

$$0 = \sum_v \langle \chi_v, \alpha_v \rangle.$$

However from Lemma 1 we see that in the sum above we have nontrivial contribution from a place $v \nmid \ell$ (and where E has good reduction) only if ℓ divides $\#\tilde{E}(\mathbb{F}_v)$. Since $\#\tilde{E}(\mathbb{F}_v)$ is of the order $\#\mathbb{F}_v$, which is the norm of v , we see that the finite places of good reduction that are involved in the sum are all of large norm. As for the bad reduction places, the heuristic assumption that we discussed just before Proposition 4 implies that these will not play any role in this sum, since it will be likely that $E(K_v)/\ell = 0$ for such places v , because v is of small norm. This explains why the index calculus method is lacking in the case of the elliptic curve discrete logarithm problem.

7. Characterization of ramification signature

Let K, ℓ, p, u, v, S be as in Proposition 2.

Let $g \in \mathbb{Z}$ so that $g \bmod p$ generates the multiplicative group of \mathbb{F}_p . Let w be the place of $K(\mu_\ell)$ over v such that $g^{\frac{p-1}{\ell}} \equiv \zeta \pmod{w}$.

Let $M = K_S$ be the cyclic extension corresponding to $H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$. Suppose $\chi \in H^1(G_S, \mathbb{Z}/\ell\mathbb{Z})$ is nontrivial. Then χ corresponds to some $A \in K(\mu_\ell)$ through $H^1(K(\mu_\ell), \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(K(\mu_\ell), \mu_\ell) \cong K(\mu_\ell)^*/K(\mu_\ell)^{\ast\ell}$, such that $M(\mu_\ell) = K(\mu_\ell)(A^{\frac{1}{\ell}})$, and for all σ in the absolute Galois group of K , $\chi(\sigma) = i$ iff $\sigma(A^{\frac{1}{\ell}})/A^{\frac{1}{\ell}} = \zeta^i$.

The following proposition provides a concrete characterization of the signature of χ .

PROPOSITION 6. *If we identify $K(\mu_\ell)_w$ with \mathbb{Q}_p and K_u with \mathbb{Q}_ℓ , then $A \sim^\ell p^m$ in \mathbb{Q}_p^{ur} where $m = \sigma_v(\chi) = \langle \chi_v, g_v \rangle$, and $A \sim^\ell \zeta^n$ in $\mathbb{Q}_\ell(\mu_\ell)^{ur}$ where $n = \sigma_u(\chi) = \langle \chi_u, 1 + \ell_u \rangle$.*

The rest of this section is devoted to the proof of this proposition. We set some notation first. For any local field L , let L^{ur} denote the maximal unramified extension over L . For any place ν of a number field K , let θ_ν denote the local Artin map,

$$\theta_\nu : K_\nu^* \rightarrow G_\nu^{ab},$$

where G_ν^{ab} denotes the Galois group of the maximal abelian extension of K_ν . For $a, b \in K(\mu_\ell)$ and ν a prime of $K(\mu_\ell)$, we have

$$\alpha^{\theta_\nu(b)} = (a, b)_\nu \alpha$$

where $\alpha^l = a$, and $(a, b)_\nu$ denotes the local norm residue symbol (see p. 351 of [CF]).

LEMMA 3. $\sigma_u(\chi) = \langle \chi_u, 1 + \ell_u \rangle = \chi_u(\theta_u(1 + \ell))$ and $\sigma_v(\chi) = \langle \chi_v, g_v \rangle = \chi_v(\theta_v(g))$

Proof This follows directly from [S1] Chapter XIV Proposition 3.

Proof of Proposition 6 Suppose v' is a place of $K(\mu_\ell)$ such that $v'|v$. Then $d < \chi_v, b_v \rangle = \langle \chi_{v'}, b_{v'} \rangle$ where $d = [K(\mu_\ell)_{v'} : K_v]$ (see [S], Proposition 7 of Ch. XIII). Moreover $\langle \chi_{v'}, b_{v'} \rangle = \chi_{v'}(\theta_{v'}(b)) = i$ iff $(A, b)_{v'} = \zeta^i$. Identifying i with ζ^i , we may write

$$d \langle \chi_v, b_v \rangle = \langle \chi_{v'}, b_{v'} \rangle = (A, b)_{v'}$$

We analyze the situation at p and ℓ separately.

(I) At p : $\mathbb{Q}_p^*/\ell = \mu_\ell \times \langle p \rangle / \ell$. So under the identification of $K(\mu_\ell)_w$ with \mathbb{Q}_p , $A = up^{w(A)}$ where $u^\ell = 1$, and $e < \ell$. Since $\mathbb{Q}_p(u^{\frac{1}{\ell}})/\mathbb{Q}_p$ is unramified, $A \sim^\ell p^{w(A)}$ in \mathbb{Q}_p^{ur} .

Let $\chi \in H^1(K, \mathbb{Z}/\ell\mathbb{Z})$

$$\begin{aligned} \langle \chi_w, g_w \rangle &= (A, g)_w = -(g, A)_w \\ (g, A)_w &= i \text{ iff } \zeta^i = \left(\frac{g}{w}\right)^{w(A)} \\ \left(\frac{g}{w}\right) &\equiv g^{\frac{Nw-1}{\ell}} \pmod{P_w} \\ &\equiv g^{\frac{p-1}{\ell}} \pmod{P_w} \\ &\equiv \zeta \pmod{P_w} \end{aligned}$$

Therefore, $(g, A)_w = w(A)$. Consequently,

$$\langle \chi_v, g_v \rangle = \langle \chi_w, g_w \rangle = -(g, A)_w = -w(A).$$

(II) At ℓ : Denote by u' the place of $K(\mu_\ell)$ over u . We have

$$(\ell - 1) \langle \chi_u, 1 + \ell_u \rangle = \langle \chi_{u'}, 1 + \ell_{u'} \rangle = (A, 1 + \ell)_{u'}.$$

We verify below that $(A, 1 + \ell)_{u'} = n$. Then we can conclude that

$$\sigma_u(\chi) = \langle \chi_u, 1 + \ell_u \rangle = -n.$$

There is a ramified extension of degree ℓ over \mathbb{Q}_ℓ , namely, the subextension M_1 of $\mathbb{Q}_\ell(\zeta_{\ell^2})$ of degree ℓ over \mathbb{Q}_ℓ . Let ψ be the ramified character in $H^1(\mathbb{Q}_\ell, \mathbb{Z}/\ell\mathbb{Z})$ whose restriction to $H^1(\mathbb{Q}_\ell(\zeta), \mathbb{Z}/\ell\mathbb{Z})$ corresponds to the class of ζ under the isomorphism $H^1(\mathbb{Q}_\ell(\zeta), \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(\mathbb{Q}_\ell(\zeta), \mu_\ell) \cong \mathbb{Q}_\ell(\zeta)^*/\ell$. Then the kernel of ψ

corresponds to M_1 .

There is an unramified extension N of degree ℓ over \mathbb{Q}_ℓ (an Artin-Schrier extension). Let $N(\zeta) = \mathbb{Q}_\ell(\zeta)(\beta^{\frac{1}{\ell}})$ with $\beta \in \mathbb{Q}_\ell(\zeta)^*$. Let φ be the unramified character in $H^1(\mathbb{Q}_\ell, \mathbb{Z}/\ell\mathbb{Z})$ whose restriction in $H^1(\mathbb{Q}_\ell(\zeta), \mathbb{Z}/\ell\mathbb{Z})$ corresponds to the class of β under the isomorphism $H^1(\mathbb{Q}_\ell(\zeta), \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(\mathbb{Q}_\ell(\zeta), \mu_\ell) \cong \mathbb{Q}_\ell(\zeta)^*/\ell$. Note that since N is unramified, $\beta^{\frac{1}{\ell}} \in \mathbb{Q}_\ell(\zeta)^{ur}$.

From Tate local duality we see that $H^1(\mathbb{Q}_\ell, \mathbb{Z}/\ell\mathbb{Z})$ has the same dimension as \mathbb{Q}_ℓ^*/ℓ . The latter is isomorphic to $\mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/\ell\mathbb{Z}$. So the dimension of $H^1(\mathbb{Q}_\ell, \mathbb{Z}/\ell\mathbb{Z})$ is two. Since the two characters ψ and φ are independent, one being ramified and the other not, they form a basis of $H^1(\mathbb{Q}_\ell, \mathbb{Z}/\ell\mathbb{Z})$ over $\mathbb{Z}/\ell\mathbb{Z}$. It follows that every character in $H^1(\mathbb{Q}_\ell, \mathbb{Z}/\ell\mathbb{Z})$ is of the form $a\psi + b\varphi$ with $a, b \in \mathbb{Z}/\ell\mathbb{Z}$. The restriction of $a\psi + b\varphi$ in $H^1(\mathbb{Q}_\ell(\zeta), \mathbb{Z}/\ell\mathbb{Z})$ corresponds to the class of $\rho = \zeta^a \beta^b$, and gives rise to a cyclic extension M' of degree ℓ over \mathbb{Q}_ℓ with $M'(\zeta) = \mathbb{Q}_\ell(\zeta)(\rho^{\frac{1}{\ell}})$. Note that $\rho \sim^\ell \zeta^i$ in $\mathbb{Q}_\ell(\zeta)^{ur}$ as $\beta^{\frac{1}{\ell}} \in \mathbb{Q}_\ell(\zeta)^{ur}$.

Since φ is unramified and $1 + \ell$ is a unit,

$$\langle \varphi, 1 + \ell \rangle = 0.$$

So

$$\langle a\psi + b\varphi, 1 + \ell \rangle = a \langle \psi, 1 + \ell \rangle = a(\zeta, 1 + \ell).$$

Since $1 + \ell = \eta_{\ell-1}\xi$ with $\xi \equiv 1 \pmod{\lambda^\ell}$,

$$(\eta_1, 1 + \ell) = (\eta_1, \eta_{\ell-1}\xi) = (\eta_1, \eta_{\ell-1})$$

$$(\eta_1, \eta_{\ell-1}) = (\eta_1, \eta_\ell) + (\eta_\ell, \eta_1) - (\ell - 1)(\eta_\ell, \lambda) = 1.$$

([CF] p.354; our symbol is written additively.)

Therefore, $\langle a\psi + b\varphi, 1 + \ell \rangle = a$.

The restriction of χ_u corresponds to $a\psi + b\varphi$, with $a, b \in \mathbb{Z}/\ell\mathbb{Z}$, under the isomorphism between $H^1(K_u, \mathbb{Z}/\ell\mathbb{Z})$ and $H^1(\mathbb{Q}_\ell, \mathbb{Z}/\ell\mathbb{Z})$. From the discussion above, $A \sim^\ell \zeta^a \beta^b$ under the identification of $K(\mu_\ell)_u$ with $\mathbb{Q}_\ell(\mu_\ell)$, and $A \sim^\ell \zeta^a$ in $\mathbb{Q}_\ell(\mu_\ell)^{ur}$.

We have

$$(\ell - 1) \langle \chi_u, 1 + \ell_u \rangle = \langle \chi_{u'}, 1 + \ell_{u'} \rangle = \langle a\psi + b\varphi, 1 + \ell \rangle = a.$$

So

$$n = \sigma_u(\chi) = \langle \chi_u, 1 + \ell_u \rangle = -a$$

where $A \sim^\ell \zeta^a$ in $\mathbb{Q}_\ell(\mu_\ell)^{ur}$.

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